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A Scaling Method for Deriving Kinetic Equations from the BBGKY Hierarchy

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On the basis of the scale covariance of correlation functions under a coarse-graining in space and time, the Boltzmann equation for neutral gases, the Balescu–Lenard–Boltzmann–Landau equation for dilute plasmas, and linear equations for the variances of fluctuations are derived from the BBGKY hierarchy equations with no short-range correlations at the initial time. This is done by using Mori's scaling method in an extended form. Thus it is shown that the scale invariance of macroscopic features affords a useful principle in nonequilibrium statistical mechanics. It is also shown that there exist *two kinds* of correlation functions, one describing the interlevel correlations of the kinetic level with its sublevels and the other representing the fluctuations in the kinetic level.

KEY WORDS: Coarse-graining in space and time; scale invariance; scale covariance; BBGKY hierarchy; kinetic scaling; interlevel correlations; fluctuations in μ space.

1. INTRODUCTION

Coarse-graining in space and time is essential for deriving macroscopic equations from the statistical mechanical standpoint. It has been recently realized that this coarse-graining in space and time can be formulated exactly by means of a projector method for reducing variables and a scaling method for reducing processes.(1) The fundamental concept then introduced is the principle of *macroscopic scale invariance*: that a macroscopic equation must be invariant under the scaling for a relevant coarse-graining in space and time. In previous papers kinetic equations for a spatially coarse-grained particle density in μ phase space of neutral gases and dilute plasmas and for its fluctuations have been derived from this point of view.(2,3)

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Since the monumental work by Bogoliubov[4] and Kirkwood[5] in 1946, various attempts to obtain kinetic equations from the Liouville equation or equivalent BBGKY hierarchy equations have been made under various specific assumptions.[6] In those attempts, however, coarse-graining in space has not been carried out explicitly. Coarse-graining in time is closely related to this coarse-graining in space, and these two coarse-grainings must be carried out coherently. This can be formulated by means of a scaling method proposed by Mori.[1] Indeed, in the present paper, we shall show that kinetic equations can be derived from the BBGKY hierarchy equations by requiring that the kinetic equations are invariant under the scaling for the kinetic coarse-graining in space and time. Thus we can avoid Bogoliubov's functional Ansatz, Kirkwood's time smoothing, and the factorization Ansatz which conventional theories employ.

In the present paper we shall also study the fluctuations around the kinetic equations. The fluctuations around the Boltzmann equation have been studied in a discrete kinetic model without spatial inhomogeneity by van Kampen with the aid of the system-size expansion.[7] We shall study them from first principles and for more realistic molecular models with the aid of the scaling method. Thus it will turn out that there exist *two kinds* of correlation functions, one describing the contributions of the sublevel processes and the other representing the fluctuations of the kinetic processes.

Let us consider a neutral gas of identical particles with mass m and a small mean particle density c, and a dilute gas of electrons with charge $-e$ in a neutralizing, smeared-out background of positive charge with charge density ce. The neutral gas has two characteristic lengths, the linear force range r_0 and the mean free path $l_f \equiv 1/cr_0^2$, and the corresponding time scales, the mean duration of a collision $\tau_0 \equiv r_0/(k_BT/m)^{1/2}$ and the mean free time $\tau_f \equiv l_f/(k_BT/m)^{1/2}$, T being the temperature. The electron plasma has three characteristic lengths, the Landau cutoff $r_0 \equiv 4\pi e^2/k_BT$, the Debye length $\lambda_D \equiv (k_BT/4\pi e^2 c)^{1/2}$, and the mean free path $l_f \equiv 1/cr_0^2$, and the corresponding time scales, $\tau_0 \equiv \epsilon/\omega_p$, $\tau_D \equiv 1/\omega_p$, and $\tau_f \equiv 1/\epsilon\omega_p$, where $\omega_p \equiv (4\pi e^2 c/m)^{1/2}$ is the plasma frequency and $\epsilon \equiv r_0/\lambda_D = \lambda_D/l_f$ is the plasma parameter. In the kinetic region, the length cutoff b and the time cutoff t_c are set as

$$l_f \gg b \gg l_m, \qquad \tau_f \gg t_c \gg \tau_m \tag{1}$$

where the sublevel characteristic length and time (l_m, τ_m) represent (r_0, τ_0) in the case of the neutral gas and $(\lambda_D, 1/\omega_p)$ in the case of the electron plasma.

The slowly varying degrees of freedom of interest are coupled to the rapidly varying processes whose length scales are smaller than b or whose time scales are shorter than t_c. In order to take into account this coupling fully, we eliminate the rapidly varying processes and derive a closed equation of motion for the slowly varying degrees of freedom. Thus we obtain a kinetic equation renormalized by the rapidly varying processes. A general method for such an elimination is provided by the projector method and the scaling method.[1,2] In the present paper, however, we shall show that such an elimination can be carried out by just applying Mori's scaling method to the BBGKY hierarchy equations without introducing the cutoffs (b, t_c) explicitly.

In Section 2 we summarize basic equations. In Sections 3 and 4 the kinetic processes of neutral gases and dilute plasmas are explored. Section 5 is devoted to a summary and remarks.

2. BASIC EQUATIONS

Starting from the Liouville equation, we can obtain the BBGKY hierarchy equations in the thermodynamic limit[4]:

$$\left[\partial_t + \sum_{i=1}^{s} L(i) - \sum_{1\leqslant i<j}^{s} \theta_{ij}\right] f_s(1,\dots, s; t)$$

$$= c \sum_{i=1}^{s} \int \theta_{i,s+1} f_{s+1}(1,\dots, s+1; t)\, d(s+1) \tag{2}$$

where $\partial_t \equiv \partial/\partial t$, $i \equiv (\mathbf{r}_i, \mathbf{p}_i)$ represents the position and momentum of the ith particle, and c denotes the mean particle density. In (2), $L(i)$ and θ_{ij} are the differential operators defined by

$$L(i) \equiv (\mathbf{p}_i/m)\cdot(\partial/\partial \mathbf{r}_i) + (\partial/\partial \mathbf{p}_i)\cdot \mathbf{K}(\mathbf{p}_i, \mathbf{r}_i) \tag{3}$$

$$\theta_{ij} \equiv [\partial V(\mathbf{r}_{ij})/\partial \mathbf{r}_{ij}]\cdot[(\partial/\partial \mathbf{p}_i) - (\partial/\partial \mathbf{p}_j)] = \theta_{ji} \tag{4}$$

where $\mathbf{r}_{ij} \equiv \mathbf{r}_i - \mathbf{r}_j$, and $\mathbf{K}(\mathbf{p}_i, \mathbf{r}_i)$ and $V(\mathbf{r}_{ij})$ are the external field and the intermolecular potential, respectively. Here the distribution functions are normalized in such a way that $(1/\Omega)\int f_1(1)\, d(1) = 1$ and

$$(1/\Omega) \int f_{s+1}(1,\dots, s+1)\, d(s+1) = f_s(1,\dots, s),$$

Ω being the volume of the system. Let us introduce the correlation functions $G(1, 2)$, $H(1, 2, 3)$, $I(1, 2, 3, 4)$,... through the Ursell–Mayer procedure:

$$f_2(1, 2) = f(1)f(2) + G(1, 2) \tag{5a}$$

$$f_3(1, 2, 3) = f(1)f(2)f(3) + f(1)G(2, 3) + f(2)G(3,1) + f(3)G(1, 2) + H(1, 2, 3) \tag{5b}$$

$$\begin{aligned} f_4(1,2,3,4) = {} & f(1)f(2)f(3)f(4) + f(1)H(2,3,4) \\ & + f(2)H(3,4,1) + f(3)H(4,1,2) \\ & + f(4)H(1,2,3) + f(1)f(2)G(3,4) \\ & + f(3)f(4)G(1,2) + f(2)f(3)G(4,1) \\ & + f(4)f(1)G(2,3) + f(1)f(3)G(2,4) \\ & + f(2)f(4)G(1,3) + I(1,2,3,4) \end{aligned} \tag{5c}$$

and so on, where $f(1) \equiv f_1(1)$. Then the BBGKY hierarchy equations are rewritten as

$$[\partial_t + L(1)]f(1;t) = c\int d(2)\theta_{12}[f(1;t)f(2;t) + G(1,2;t)] \tag{6}$$

$$\begin{aligned} & [\partial_t + L(1) + L(2) - \theta_{12}]G(1,2;t) \\ & \quad = \theta_{12}f(1;t)f(2;t) \\ & \qquad + c\int d(3)\,[\theta_{13}f(1;t)G(2,3;t) + \theta_{23}f(2;t)G(3,1;t)] \\ & \qquad + c\int d(3)\,(\theta_{13} + \theta_{23})[f(3;t)G(1,2;t) + H(1,2,3;t)] \end{aligned} \tag{7}$$

$$\begin{aligned} & [\partial_t + L(1) + L(2) + L(3) - (\theta_{12} + \theta_{23} + \theta_{31})]H(1,2,3;t) \\ & \quad = (\theta_{12} + \theta_{13})f(1;t)G(2,3;t) + (\theta_{21} + \theta_{23})f(2;t)G(3,1;t) \\ & \qquad + (\theta_{31} + \theta_{32})f(3;t)G(1,2;t) \\ & \qquad + c\int d(4)\,\{(\theta_{24} + \theta_{34})G(1,4;t)G(2,3;t) \\ & \qquad + (\theta_{14} + \theta_{34})G(2,4;t)G(1,3;t) + (\theta_{14} + \theta_{24})G(3,4;t)G(1,2;t) \\ & \qquad + \theta_{14}f(1;t)H(2,3,4;t) + \theta_{24}f(2;t)H(3,4,1;t) \\ & \qquad + \theta_{34}f(3;t)H(4,1,2;t) + (\theta_{14} + \theta_{24} + \theta_{34})[f(4,t)H(1,2,3;t) \\ & \qquad + I(1,2,3,4;t)]\} \end{aligned} \tag{8}$$

Equation (7) is integrated to give

$$\begin{aligned} & G(1,2;t) \\ & \quad = S_{-t}^{(2)}(1,2)G(1,2;0) + \int_0^t ds\, S_{-s}^{(2)}(1,2) \\ & \qquad \times [\theta_{12}f(1;t-s)f(2;t-s) + c\int d(3)\,\{\theta_{13}f(1;t-s)G(2,3;t-s) \\ & \qquad + \theta_{23}f(2;t-s)G(1,3;t-s) \\ & \qquad + (\theta_{13} + \theta_{23})[f(3;t-s)G(1,2;t-s) + H(1,2,3;t-s)]\}] \end{aligned} \tag{9}$$

where $S_t^{(2)}(1,2) \equiv \exp\{t[L(1) + L(2) - \theta_{12}]\}$.

The variance matrix of fluctuations is given by

$$\chi(1, 2; t) = (1/c)\,\delta(1 - 2)f(1; t) + G(1, 2; t) \tag{10}$$

where $\delta(1 - 2) \equiv \delta(\mathbf{p}_1 - \mathbf{p}_2)\,\delta(\mathbf{r}_1 - \mathbf{r}_2)$. Equations (6) and (7) lead to

$$\begin{aligned}
&[\partial_t + L(1) + L(2) - \theta_{12}]\chi(1, 2; t) \\
&\quad = -(1/c)\theta_{12}\,\delta(1 - 2)f(1; t) + \delta(1 - 2)\int d(3)\,\theta_{13}G(1, 3; t) \\
&\qquad + c\int d(3)\,[\theta_{13}f(1; t)\chi(2, 3; t) + \theta_{23}f(2; t)\chi(3, 1; t)] \\
&\qquad + c\int d(3)\,(\theta_{13} + \theta_{23})[f(3; t)\chi(1, 2; t) + H(1, 2, 3; t)]
\end{aligned} \tag{11}$$

Equation (8) is integrated to give

$$\begin{aligned}
&H(1, 2, 3; t) \\
&\quad = S_{-t}^{(3)}(1, 2, 3)H(1, 2, 3; 0) + \int_0^t ds\, S_{-s}^{(3)}(1, 2, 3)\Big[(\theta_{12} + \theta_{13}) \\
&\qquad \times f(1; t - s)[\chi(2, 3; t - s) - (1/c)\,\delta(2 - 3)f(2; t - s)] \\
&\qquad + (\theta_{21} + \theta_{23})f(2; t - s)[\chi(1, 3; t - s) - (1/c)\,\delta(1 - 3)f(1; t - s)] \\
&\qquad + (\theta_{31} + \theta_{32})f(3; t - s)[\chi(1, 2; t - s) - (1/c)\,\delta(1 - 2)f(1; t - s)] \\
&\qquad + c\int d(4)\,\{(\theta_{24} + \theta_{34})G(1, 4; t - s)[\chi(2, 3; t - s) \\
&\qquad - (1/c)\,\delta(2 - 3)f(2; t - s)] \\
&\qquad + (\theta_{14} + \theta_{34})G(2, 4; t - s)[\chi(1, 3; t - s) - (1/c)\,\delta(1 - 3)f(1; t - s)] \\
&\qquad + (\theta_{14} + \theta_{24})G(3, 4; t - s)[\chi(1, 2; t - s) - (1/c)\,\delta(1 - 2)f(1; t - s)] \\
&\qquad + \theta_{14}f(1; t - s)H(2, 3, 4; t - s) + \theta_{24}f(2; t - s)H(3, 4, 1; t - s) \\
&\qquad + \theta_{34}f(3; t - s)H(4, 1, 2; t - s) + (\theta_{14} + \theta_{24} + \theta_{34}) \\
&\qquad \times [f(4; t - s)H(1, 2, 3; t - s) + I(1, 2, 3, 4; t - s)]\}\Big]
\end{aligned} \tag{12}$$

where $S_t^{(3)}(1, 2, 3) \equiv \exp\{t[L(1) + L(2) + L(3) - \theta_{12} - \theta_{23} - \theta_{31}]\}$.

3. THE KINETIC REGION OF NEUTRAL GASES

In this section we assume that the intermolecular force is of short range, with a linear force range r_0. Then, according to Mori's scaling method,[1] the kinetic processes characterized by l_f and τ_f can be extracted by the scaling

$$l_f \to L l_f, \qquad r_0 \to r_0, \qquad c \to c/L \tag{13a}$$

$$\mathbf{r}_1 \to L\mathbf{r}_1, \qquad t \to Lt \tag{13b}$$

with $L \gg 1$. The mass, momenta, and temperature are unchanged. The scaling (13b) is obtained from $l_f \to L l_f$ and $\tau_f \to L\tau_f$. The requirement of the scale invariance of kinetic equations leads to the scaled form

$$f(1; t) = \tilde{F}(\mathbf{p}_1, \mathbf{r}_1/l_f; t/l_f) \tag{14}$$

where the scaling exponent has been determined from the normalization condition. The scale invariance also leads to the following scaled forms for G, H, and higher order correlation functions describing the contributions of the collision processes characterized by r_0 and τ_0:

$$G(1, 2; t) = (r_0/l_f)^{\mu_1}\tilde{G}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{r}_1/l_f, \mathbf{r}_{21}/r_0; t/l_f, t/r_0) \tag{15a}$$

$$H(1, 2, 3; t) = (r_0/l_f)^{\mu_2}\tilde{H}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{r}_1/l_f, \mathbf{r}_{21}/r_0, \mathbf{r}_{31}/r_0; t/l_f, t/r_0) \tag{15b}$$

where $\tilde{G}$ and $\tilde{H}$ are scale invariants. The exponents μ_1 and μ_2 represent the magnitudes of the correlation functions, and (9) and (12) lead to $\mu_1 = \mu_2 = 0$. Applying the kinetic scaling (13) to (6) and (9), we thus obtain for large L

$$[\partial_t + L(1)]f(1; t) = c \int d(2)\, \theta_{12}[f(1; t)f(\mathbf{p}_2, \mathbf{r}_1; t) + G(1, 2; t)] \tag{16}$$

$$\begin{aligned} G(1, 2; t) = &\int_0^\infty ds \exp[-s(\mathbf{g}_{21}\cdot\partial/\partial\mathbf{r}_{21} - \theta_{12})]\theta_{12}f(1; t)f(\mathbf{p}_2, \mathbf{r}_1; t) \\ &+ \exp\{-t[(\mathbf{p}_1/m)\cdot\partial/\partial\mathbf{r}_1 \\ &+ (\partial/\partial\mathbf{p}_1)\cdot\mathbf{K}(\mathbf{p}_1, \mathbf{r}_1) + (\partial/\partial\mathbf{p}_2)\cdot\mathbf{K}(\mathbf{p}_2, \mathbf{r}_1)]\} \\ &\times \lim_{L\to\infty} \exp[-Lt(\mathbf{g}_{21}\cdot\partial/\partial\mathbf{r}_{21} - \theta_{12})]G(1, 2; 0) \end{aligned} \tag{17}$$

where $\mathbf{g}_{21} \equiv (\mathbf{p}_2 - \mathbf{p}_1)/m$, and $\mathbf{r}_{21}$ and s have been kept unchanged in accordance with r_0, $\tau_0 \to r_0$, τ_0. Balancing the two terms in $L(1)$, we obtain

$$\mathbf{K}(\mathbf{p}_1, \mathbf{r}_1) = (r_0/l_f)\tilde{\mathbf{K}}(\mathbf{p}_1, \mathbf{r}_1/l_f) \tag{18}$$

where $\tilde{\mathbf{K}}$ is a scale invariant. Let us assume $\tilde{G}(1, 2; 0) = 0$ at the initial time. Then we obtain the nonlinear kinetic equation

$$[\partial_t + L(1)]f(1; t) = c \int d(2)\, T(1, 2)f(1; t)f(\mathbf{p}_2, \mathbf{r}_1; t) \tag{19}$$

where

$$T(1, 2) \equiv \theta_{12}\mathscr{L}_{\infty}(1, 2) \tag{20a}$$

$$\begin{aligned}\mathscr{L}_t(1, 2) &\equiv \exp[-t(\mathbf{g}_{21}\cdot\partial/\partial\mathbf{r}_{21} - \theta_{12})]\exp(t\mathbf{g}_{21}\cdot\partial/\partial\mathbf{r}_{21}) \\ &= 1 + \int_0^t ds\,\exp[-s(\mathbf{g}_{21}\cdot\partial/\partial\mathbf{r}_{21} - \theta_{12})]\theta_{12}\exp(s\mathbf{g}_{21}\cdot\partial/\partial\mathbf{r}_{21})\end{aligned} \tag{20b}$$

The rhs of (19) can be transformed into the usual Boltzmann collision integral.[8] Thus we obtain the Boltzmann equation for the spatially inhomogeneous dilute gas in an external field.

Next let us consider the fluctuations of the kinetic processes. The scale covariance of variances and correlations leads to the scaled forms

$$\chi(1, 2; t) = (r_0/l_f)^{\nu_0}\tilde{\chi}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{r}_1/l_f, \mathbf{r}_{21}/l_f; t/l_f) \tag{21a}$$

$$\begin{aligned}&H(1, 2, 3; t) \\ &\quad = (r_0/l_f)^{\nu_1}\tilde{H}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{r}_1/l_f, \mathbf{r}_{21}/l_f, \mathbf{r}_{31}/r_0; t/l_f, t/r_0)\end{aligned} \tag{21b}$$

$$\begin{aligned}&I(1, 2, 3, 4; t) \\ &\quad = (r_0/l_f)^{\nu_2}\tilde{I}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4, \mathbf{r}_1/l_f, \mathbf{r}_{21}/l_f, \mathbf{r}_{31}/r_0, \mathbf{r}_{41}/r_0; t/l_f, t/r_0)\end{aligned} \tag{21c}$$

for the molecular configurations where molecules 3 and 4 locate near 1. Here $\tilde{\chi}$, $\tilde{H}$, and $\tilde{I}$ are scale invariants. This scale covariance results from the scale invariance of the probability for fluctuations.[1] We assume $\tilde{H}(1, 2, 3; 0) = 0$ at the initial time. Then, applying the scaling (13) and $\mathbf{r}_{21} \to L\mathbf{r}_{21}$ to (12), we obtain, for $d > 1$,

$$\begin{aligned}&\int d(3)\,\theta_{13}H(1, 2, 3; t) \\ &\quad \to L^{-d+1}\int d(3)\,\theta_{13}\int_0^{\infty} ds\,\exp[-s(\mathbf{g}_{31}\cdot\partial/\partial\mathbf{r}_{31} - \theta_{13})] \\ &\qquad \times \theta_{13}\{f(1; t)[\chi(2, \mathbf{p}_3, \mathbf{r}_1; t) - (1/c)\,\delta(\mathbf{p}_2 - \mathbf{p}_3)\,\delta(\mathbf{r}_2 - \mathbf{r}_1)f(2; t)] \\ &\qquad + f(\mathbf{p}_3, \mathbf{r}_1; t)[\chi(1, 2; t) - (1/c)\,\delta(1 - 2)f(1; t)]\}\end{aligned} \tag{22}$$

and $\nu_0 = \nu_1 = \nu_2 = d - 1$, where d is the spatial dimensionality and we have used the fact that $\theta_{12} \to L^{-(1+\phi)}\theta_{12}$ with $\phi > 0$. Similarly, from (11) we obtain

$$\begin{aligned}&[\partial_t + L(1) + L(2)]\chi(1, 2; t) \\ &\quad = \delta(1 - 2)\int d(3)\,T(1, 3)f(1; t)f(\mathbf{p}_3, \mathbf{r}_1; t) \\ &\qquad + c\int d(3)\,[\theta_{13}f(1; t)\chi(2, \mathbf{p}_3, \mathbf{r}_1; t) + \theta_{23}f(2; t)\chi(1, \mathbf{p}_3, \mathbf{r}_2; t) \\ &\qquad + \theta_{13}f(\mathbf{p}_3, \mathbf{r}_1; t)\chi(1, 2; t) + \theta_{23}f(\mathbf{p}_3, \mathbf{r}_2; t)\chi(1, 2; t) \\ &\qquad + (\theta_{13} + \theta_{23})H(1, 2, 3; t)]\end{aligned} \tag{23}$$

Inserting (22) into (23), we obtain the linear variance equation

$$[\partial_t + L(1) + L(2)]\chi(1, 2; t)$$
$$= c \int d(3)\{T(1, 3)[f(1; t)\chi(2, \mathbf{p}_3, \mathbf{r}_1; t) + f(\mathbf{p}_3, \mathbf{r}_1; t)\chi(1, 2; t)]$$
$$+ T(2, 3)[f(2; t)\chi(1, \mathbf{p}_3, \mathbf{r}_2; t)$$
$$+ f(\mathbf{p}_3, \mathbf{r}_2; t)\chi(1, 2; t)]\} + 2cE_{1,2}(f) \tag{24a}$$

$$E_{1,2}(f) \equiv (1/2c)\,\delta(1 - 2) \int d(3)\, T(1, 3) f(1; t) f(\mathbf{p}_3, \mathbf{r}_1; t)$$
$$- (1/2c) \int d(3)\{T(1, 3)[f(1; t)\,\delta(\mathbf{p}_2 - \mathbf{p}_3)\,\delta(\mathbf{r}_2 - \mathbf{r}_1) f(2; t)$$
$$+ f(\mathbf{p}_3, \mathbf{r}_1; t)\,\delta(1 - 2) f(1; t)]$$
$$+ T(2, 3)[f(2; t)\,\delta(\mathbf{p}_1 - \mathbf{p}_3)\,\delta(\mathbf{r}_1 - \mathbf{r}_2) f(1; t)$$
$$+ f(\mathbf{p}_3, \mathbf{r}_2; t)\,\delta(1 - 2) f(1; t)]\} \tag{24b}$$

where we have used the fact that $\int d\mathbf{r}_{31}\, \theta_{13} = 0$.

If $d > 1$, then $\chi(1, 2)/f(1)f(2) \sim (r_0/l_f)^{d-1} \ll 1$ and the fluctuations are very small compared to the systematic part $f(1; t)$. Then the fluctuations are described by a Gaussian Markov process specified by the variances $\chi(1, 2)$ and the diffusion coefficients $E_{1,2}(f)$.[1,2] Thus if $d > 1$, then the kinetic processes are completely determined by $f(1; t)$ and $\chi(1, 2; t)$.

4. THE KINETIC REGION OF PLASMAS

In this section we consider the kinetic processes of a dilute classical electron plasma. We assume that the plasma parameter $\epsilon \equiv 1/c\lambda_D^3$ is small so that $r_0/\lambda_D = \lambda_D/l_f = \epsilon \ll 1$. The scaling (13a) leads to

$$l_f \to L l_f, \qquad \lambda_D \to \sqrt{L}\,\lambda_D, \qquad r_0 \to r_0, \qquad c \to c/L, \qquad \epsilon \to \epsilon/\sqrt{L} \tag{25}$$

where charge e, mass m, and temperature T are unchanged.

Let us proceed similarly to the previous section. In the kinetic region characterized by l_f and τ_f, we have the scaled form (14); namely,

$$f(1; t) = \tilde{F}(\mathbf{p}_1, \mathbf{r}_1/l_f; t/l_f) \tag{26a}$$

There exist two different sublevels, specified by (r_0, τ_0) and $(\lambda_D, 1/\omega_p)$, respectively. The correlation functions that describe the interlevel correlations of the kinetic level (l_f, τ_f) with the two sublevels are indicated by subscripts 1 and 2, respectively. Then the scale covariance leads to

$$G_1(1, 2; t) = \epsilon^{\mu_1}\tilde{G}_1(\mathbf{p}_1, \mathbf{p}_2, \mathbf{r}_1/l_f, \mathbf{r}_{21}/r_0; t/l_f, t/r_0) \tag{26b}$$

$$G_2(1, 2; t) = \epsilon^{\mu_2}\tilde{G}_2(\mathbf{p}_1, \mathbf{p}_2, \mathbf{r}_1/l_f, \mathbf{r}_{21}/\lambda_D; t/l_f, t/\lambda_D) \tag{26c}$$

and similar scaled forms for H. In order to separate the contributions of the two sublevels, we introduce a length cutoff a satisfying $r_0 < a \ll \lambda_D$ and rewrite (6) as

$$\begin{aligned}[\partial_t + L(1)]f(1;t) &= c\int d(2)\,\theta_{12}f(1;t)f(2;t) \\ &\quad + c\Big[\int_0^\infty dr_{21}\,r_{21}^2\int d\omega\int d\mathbf{p}_2\,\theta_{12}G_1(1,2;t) \\ &\quad - \int_a^\infty dr_{21}\,r_{21}^2\int d\omega\int d\mathbf{p}_2\,\theta_{12}G_1(1,2;t) \\ &\quad + \int_a^\infty dr_{21}\,r_{21}^2\int d\omega\int d\mathbf{p}_2\,\theta_{12}G_2(1,2;t)\Big] \end{aligned} \tag{27}$$

where $d\mathbf{r}_{21} \equiv r_{21}^2\,dr_{21}\,d\omega$, $r_{21} \equiv |\mathbf{r}_{21}|$. The cutoff a is set so that its scaling is $a \to a$.

We assume $G_1(1, 2; 0) = G_2(1, 2; 0) = 0$ at the initial time. Then, as was shown in Section 3, $G_1(1, 2; t)$ is given by the first term of (17), and $\mu_1 = 0$. In the third and the fourth terms of the rhs of (27) we change the integral variable as $r_{21} = \sqrt{L}r'_{21}$ and rewrite r'_{21} as r_{21}, whereas in the first and second terms we do not make a change. Then the first and second terms lead to the Boltzmann collision term with the Coulomb potential,

$$B_1(f) \equiv \int d(2)\,T(1,2)f(1;t)f(\mathbf{p}_2,\mathbf{r}_1;t) \tag{28}$$

and the third term turns out to be the Landau collision term,[9]

$$L_1(f) \equiv \int d(2)\,\theta_{12}\int_0^\infty ds\,\exp[-s\mathbf{g}_{21}\cdot\partial/\partial\mathbf{r}_{21}]\,\theta_{12}f(1;t)f(\mathbf{p}_2,\mathbf{r}_1;t) \tag{29}$$

As will be shown in Appendix A, we have $\mu_2 = 1$,

$$\begin{aligned} G_2(1,2;t) &= \int_0^\infty ds\,\exp(-s\mathbf{g}_{21}\cdot\partial/\partial\mathbf{r}_{21})\Big\{\theta_{12}f(1;t)f(\mathbf{p}_2,\mathbf{r}_1;t) \\ &\quad + c\int d(3)\,[\theta_{13}f(1;t)G_2(\mathbf{p}_2,\mathbf{p}_3,\mathbf{r}_1,\mathbf{r}_{32};t) \\ &\quad + \theta_{23}f(\mathbf{p}_2,\mathbf{r}_1;t)G_2(1,3;t)]\Big\} \end{aligned} \tag{30}$$

and hence the fourth term leads to the Balescu–Lenard collision term for the spatially inhomogeneous plasma,[10,11]

$$\begin{aligned} BL_1(f) &= 8\pi^4\int d\mathbf{q}\int d\mathbf{p}_2\,\mathbf{q}\cdot\frac{\partial}{\partial\mathbf{p}_1}\left|\frac{V_q}{\epsilon_{\mathbf{p}_1\mathbf{r}_1}(\mathbf{q},f)}\right|^2\delta(\mathbf{q}\cdot\mathbf{g}_{12}) \\ &\quad \times \mathbf{q}\cdot\left(\frac{\partial}{\partial\mathbf{p}_1} - \frac{\partial}{\partial\mathbf{p}_2}\right)f(1;t)f(\mathbf{p}_2,\mathbf{r}_1;t) \end{aligned} \tag{31}$$

where

$$\epsilon_{\mathbf{p}_1\mathbf{r}_1}(\mathbf{q},f) \equiv 1 + 8\pi^4 icV_q \int d\mathbf{p}_2\, \delta_-(\mathbf{q}\cdot\mathbf{g}_{12})\mathbf{q}\cdot(\partial/\partial\mathbf{p}_2)f(\mathbf{p}_2, \mathbf{r}_1; t) \tag{32}$$

$V_q \equiv e^2/2\pi^2 q^2$, $\delta_-(x) \equiv \delta(x) + (i/\pi)\mathscr{P}(1/x)$, and $\mathscr{P}(1/x)$ denotes the principal part of $1/x$. Thus we obtain the Balescu–Lenard–Boltzmann–Landau equation,[3,6]

$$[\partial_t + L(1)]f(1; t) = cJ_1(f) \tag{33a}$$

$$J_1(f) \equiv B_1(f) - L_1(f) + BL_1(f) \tag{33b}$$

Each of these three collision terms has a divergence difficulty. Their sum $J_1(f)$, however, is free of divergence for both close and distant collisions.[6,12,13]

In the usual cluster expansion approach to the BBGKY hierarchy of plasmas, it is assumed that $G(1, 2)/f(1)f(2) \sim \epsilon$, $H(1, 2, 3)/f(1)f(2)f(3) \sim \epsilon^2$.[6,14,15] However, this assumption is only valid for the correlation functions with a length scale of order λ_D, such as (26c). The correlation functions with length scale of order r_0, such as (26b), have a different scaling. In the present paper, we have succeeded in taking into account both types of correlation functions and thus in deriving the divergence-free kinetic equations from the BBGKY hierarchy.

Next let us consider the fluctuations. Similarly to (26), we have the following scaled forms for the electronic configurations where electron 3 is located close to 1:

$$\chi(1, 2; t) = \epsilon^{\nu_0}\tilde{\chi}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{r}_1/l_f, \mathbf{r}_{21}/l_f; t/l_f) \tag{34a}$$

$$H_1(1, 2, 3; t) = \epsilon^{\nu_1}\tilde{H}_1(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{r}_1/l_f, \mathbf{r}_{21}/l_f, \mathbf{r}_{31}/r_0; t/l_f, t/r_0) \tag{34b}$$

$$H_2(1, 2, 3; t) = \epsilon^{\nu_2}\tilde{H}_2(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{r}_1/l_f, \mathbf{r}_{21}/l_f, \mathbf{r}_{31}/\lambda_D; t/l_f, t/\lambda_D) \tag{34c}$$

where $\tilde{\chi}$, $\tilde{H}_1$, and $\tilde{H}_2$ are scale invariants. Similarly to (27), we rewrite the H term of (11) as

$$\begin{aligned}\int d(3)\, \theta_{13}H(1, 2, 3; t) = &\int_0^\infty dr_{31}\, r_{31}^2 \int d\omega \int d\mathbf{p}_3\, \theta_{13}H_1(1, 2, 3: t)\\ &- \int_{a_\circ}^\infty dr_{31}\, r_{31}^2 \int d\omega \int d\mathbf{p}_3\, \theta_{13}H_1(1, 2, 3; t)\\ &+ \int_a^\infty dr_{31}\, r_{31}^2 \int d\omega \int d\mathbf{p}_3\, \theta_{13}H_2(1, 2, 3; t)\end{aligned} \tag{35}$$

We assume $H_1(1, 2, 3; 0) = H_2(1, 2, 3; 0) = 0$ at the initial time. Then applying the scaling (13) and $\mathbf{r}_{21} \to L\mathbf{r}_{21}$ to (12), we obtain

$$H_1(1, 2, 3; t)$$
$$\to L^{-2}\int_0^\infty ds \exp[-s(\mathbf{g}_{31}\cdot\partial/\partial\mathbf{r}_{31} - \theta_{13})]\, \theta_{13}[f(1; t)\chi(2, \mathbf{p}_3, \mathbf{r}_1; t)$$
$$+ f(\mathbf{p}_3, \mathbf{r}_1; t)\chi(1, 2; t)$$
$$- (1/c)f(1; t)\, \delta(\mathbf{p}_3 - \mathbf{p}_2)\, \delta(\mathbf{r}_1 - \mathbf{r}_2)f(2; t)$$
$$- (1/c)f(\mathbf{p}_3, \mathbf{r}_1; t)\, \delta(1 - 2)f(1; t)] \tag{36}$$

and $\nu_1 = 4$ and therefore $\nu_0 = 4$. In the second and third terms of the rhs of (35), we change the integral variable as $r_{31} = \sqrt{L}r'_{31}$ and rewrite r'_{31} as r_{31}, whereas in the first term we do not make a change. Then the second term leads to

$$-L^{-2}\int d(3)\, \theta_{13}\int_0^\infty ds \exp[-s\mathbf{g}_{31}\cdot\partial/\partial\mathbf{r}_{31}]$$
$$\times\ \theta_{13}[f(1; t)\chi(2, \mathbf{p}_3, \mathbf{r}_1; t) + f(\mathbf{p}_3, \mathbf{r}_1; t)\chi(1, 2; t)$$
$$- (1/c)f(1; t)\, \delta(\mathbf{p}_3 - \mathbf{p}_2)\, \delta(\mathbf{r}_1 - \mathbf{r}_2)f(2; t)$$
$$- (1/c)f(\mathbf{p}_3, \mathbf{r}_1; t)\, \delta(1 - 2)f(1; t)]$$
$$= -L^{-2}\int d(3)\, [\delta L_1(f)/\partial f(3; t)]$$
$$\times\ [\chi(3, 2; t) - (1/c)\, \delta(3 - 2)f(2; t)] \tag{37}$$

As will be shown in Appendix B, the third term leads to

$$L^{-2}\int d(3)\, \frac{\delta BL_1(f)}{\delta f(3; t)}\left[\chi(3, 2; t) - \frac{1}{c}\, \delta(3 - 2)f(2; t)\right] \tag{38}$$

and we obtain $\nu_2 = 5$. The first term must have the same form as (22) with $d = 3$. Thus we obtain the linear variance equation

$$[\partial_t + L(1) + L(2)]\chi(1, 2; t)$$
$$= c\int d(3)\, [J_{1:3}(f)\chi(3, 2; t)$$
$$+ J_{2:3}(f)\chi(1, 3; t)] + 2cE_{1,2}(f) \tag{39a}$$
$$E_{1,2}(f) \equiv (1/2c)\, \delta(1 - 2)J_1(f)$$
$$- (1/2c)\int d(3)\, [J_{1:3}\,(f)\delta(3 - 2)f(2; t)$$
$$+ J_{2:3}\,(f)\delta(3 - 1)f(1; t)] \tag{39b}$$

where $J_{1:3}(f) \equiv \delta J_1(f)/\delta f(3)$. This has the same structure as (24). In fact, (24) is obtained from (39) by replacing $J_1(f)$ by the Boltzmann collision term. Thus it turns out that if $d > 1$, then the kinetic processes of plasmas also obey a Gaussian Markov process and are determined by $f(1; t)$ and $\chi(1, 2; t)$.

5. SUMMARY AND REMARKS

On the basis of the scale covariance of $f(1; t)$ and correlation functions under the kinetic scaling (13), we have derived the Boltzmann equation (19) for neutral gases, the Balescu–Lenard–Boltzmann–Landau equation (33) for dilute plasmas, and the evolution equations (24) and (39) for the variances of fluctuations from the BBGKY hierarchy equations with no short-range correlations at the initial time. It seems to us that the present derivation is more general and simpler than any previous derivations. Thus it would turn out that *macroscopic scale invariance* provides us with a new, useful principle in nonequilibrium statistical mechanics.

We have assumed an initial ensemble in which there are no short-range correlations. This initial ensemble ensures the removal of pathological initial conditions with zero measure, such as the time-reversed state. Since such correlations are produced by the interaction between particles, we may assume that the initial ensemble can represent any meaningful ensemble after an initial transient period of the order of the mean duration of a collision.

It has been shown that there exist *two kinds* of correlation functions, which have quite different physical meanings. One type describes the interlevel correlations of the kinetic level with its sublevels, and examples are given by (15), (21b), (21c), (26b), (26c), and (34b), (34c). The other type represents the fluctuations in the kinetic level, and examples are given by the variances (21a) and (34a). This distinction would be important for describing the hierarchical structure of dynamic processes with different characteristic length and time scales.

As has been shown in (26) and (34), different sublevels produce different interlevel correlations, which have different scaling properties. In the case of dilute plasmas, in order to separate the contributions of the two sublevels, (r_0, τ_0) and $(\lambda_D, 1/\omega_p)$, we have introduced the length cutoff a. We can use a different length cutoff a', which satisfies $r_0 \ll a' < \lambda_D$, and have the scaling $a' \to \sqrt{L}a'$. Then, instead of (27) and (35), we have

$$
\begin{aligned}
&[\partial_t + L(1)]f(1; t) - c\int d(2)\, \theta_{12} f(1; t) f(2; t) \\
&= c\left[\int_0^{a'} dr_{21}\, r_{21}^2 \int d\omega \int d\mathbf{p}_2\, \theta_{12} G_1(1, 2; t)\right. \\
&\quad - \int_0^{a'} dr_{21}\, r_{21}^2 \int d\omega \int d\mathbf{p}_2\, \theta_{12} G_2(1, 2; t) \\
&\quad \left. + \int_0^{\infty} dr_{21}\, r_{21}^2 \int d\omega \int d\mathbf{p}_2\, \theta_{12} G_2(1, 2; t)\right]
\end{aligned}
\tag{40}
$$

$$\begin{aligned}\int d(3)\,\theta_{13}H(1,2,3;t) \\ &= \int_0^{a'} dr_{31}\,r_{31}^2 \int d\omega \int d\mathbf{p}_3\,\theta_{13}H_1(1,2,3;t) \\ &\quad - \int_0^{a'} dr_{31}\,r_{31}^2 \int d\omega \int d\mathbf{p}_3\,\theta_{13}H_2(1,2,3;t) \\ &\quad + \int_0^{\infty} dr_{31}\,r_{31}^2 \int d\omega \int d\mathbf{p}_3\,\theta_{13}H_2(1,2,3;t)\end{aligned} \tag{41}$$

In the third term of the rhs of (40) and (41) we change the integral variable as $r_{21} = \sqrt{L}r'_{21}$, whereas in the first and second terms we make no change. Then in the scaling limit (13) these also lead to (33) and (39). Thus it turns out that the Boltzmann collision term comes out from G_1 and the Balescu–Lenard collision term comes out from G_2, whereas the Landau collision term is produced by either G_1 or G_2. This clarifies the origin of the dual nature of the Landau collision term.

The Vlasov equation for plasmas in the coherent region can be derived as follows. The dynamic processes characterized by λ_D and $1/\omega_p$ can be extracted by the coherent scaling[3]

$$\lambda_D \to L\lambda_D, \qquad r_0 \to r_0, \qquad c \to c/L^2, \qquad \epsilon \to \epsilon/L \tag{42a}$$

$$\mathbf{r}_1 \to L\mathbf{r}_1, \qquad t \to Lt \tag{42b}$$

with $L \gg 1$, where $\omega_p \to \omega_p/L$, $l_f \to L^2 l_f$. The scale covariance under this scaling leads to

$$f(1;t) = \tilde{F}(\mathbf{p}_1, \mathbf{r}_1/\lambda_D; t/\lambda_D) \tag{43}$$

Since the Coulomb interaction θ_{12} in (6) extends over a distance of order λ_D, we have to introduce the following two types of correlation functions:

$$G_1(1,2;t) = \tilde{G}_1(\mathbf{p}_1, \mathbf{p}_2, \mathbf{r}_1/\lambda_D, \mathbf{r}_{21}/r_0; t/\lambda_D, t/r_0) \tag{44a}$$

$$G_f(1,2;t) = \epsilon^{d-2}\tilde{G}_f(\mathbf{p}_1, \mathbf{p}_2, \mathbf{r}_1/\lambda_D, \mathbf{r}_{21}/\lambda_D; t/\lambda_D) \tag{44b}$$

for $d \geqslant 2$. The correlation function G_f represents the fluctuations in the coherent region. If $G_\alpha(1,2;0) = 0$, then G_1 is also given by the first term of (17). Since $\theta_{12} \to \theta_{12}/L^{d-1}$ for $\mathbf{r}_{12} \to L\mathbf{r}_{21}$, (9) leads, for $d \geqslant 2$, to

$$\begin{aligned}G_f(1,2;t) \to L^{-d+2} \int_0^t ds \exp\{-s[L(1) + L(2) - L^{-d+2}\theta_{12}]\} \\ &\times \{\theta_{12}f(1;t-s)f(2;t-s) \\ &+ c\int d(3)\,[\theta_{13}f(1;t-s)G_f(2,3;t-s) \\ &+ \theta_{23}f(2;t-s)G_f(3,1;t-s) \\ &+ (\theta_{13}+\theta_{23})f(3;t-s)G_f(1,2;t-s) \\ &+ L^{-d+2}(\theta_{13}+\theta_{23})H_f(1,2,3;t-s)]\}\end{aligned} \tag{45}$$

Therefore, changing the integral variable as $\mathbf{r}_{21} = L\mathbf{r}'_{21}$ in (6), we obtain, for $d > 2$,

$$[\partial_t + L(1)]f(1; t) = c \int d(2)\, \theta_{12} f(1; t) f(2; t) \tag{46}$$

This is the Vlasov equation. Thus it turns out that the streaming term $L(1)$ balances the Vlasov term in the coherent region, while it balances the BLBL collision term $cJ_1(f)$ in the kinetic region. The rhs of (46) is of order ϵ, and G_1 gives a close-collision term of order ϵ^2, and G_f produces a non-Markov interaction term of order ϵ^{d+1}. These higher order terms can be taken into account by using a multiple-time scaling.[3] In the low-dimensional plasmas ($d \leqslant 2$), the fluctuation correlation function G_f thus produces a major effect and the Vlasov equation becomes invalid.

Finally, it would be worth noting two main features of the present theory. Many theories use chopping limits, such as the Grad limit,[18] the Bogoliubov–Balsecu limit,[4,16] and the fluid limit,[14,15] where the particles are chopped so that mass m, charge e, and molecular diameter r_0 become zero with the mean particle density c increasing to infinity. Such a chopping limit, however, is not physical and cannot describe the fluctuations in the low-density limit $c \to 0$. In the present theory, microscopic quantities such as m, e, r_0, and momenta $\mathbf{p}_i$ are kept constant, and the macroscopic state parameter c is changed in accordance with the coarse-graining in space and time. Thus it has become possible to describe the fluctuations in μ space correctly. A more complete description of the kinetic processes is provided by the method of generalized Brownian motion.[1-3] However, the present approach is more convenient from the practical point of view, and its generalization to the quantal case must be easier.

APPENDIX A. DERIVATION OF (31)

Applying the scaling (13) to (9), we obtain

$$\begin{aligned} G_2(1, 2; t) \to (1/\sqrt{L}) \int_0^\infty & ds \exp(-s\mathbf{g}_{21}\cdot\partial/\partial\mathbf{r}_{21}) \\ & \times \Big\{\theta_{12} f(1; t) f(\mathbf{p}_2, \mathbf{r}_1; t) \\ & + c \int d(3)\, [\theta_{13} f(1; t) G_2(\mathbf{p}_2, \mathbf{p}_3, \mathbf{r}_1, \mathbf{r}_{32}; t) \\ & + \theta_{23} f(\mathbf{p}_2, \mathbf{r}_1; t) G_2(1, 3; t)]\Big\} \end{aligned} \tag{A.1}$$

where we have used the fact that $\int d\mathbf{r}_{31}\, \theta_{13} = 0$ and $H_2 \sim \epsilon^2$. Let us define

$$K_1(\mathbf{q}) \equiv K_{\mathbf{p}_1\mathbf{r}_1}(\mathbf{q}) \equiv (2\pi)^{-3} \int d(2)\, [\exp(i\mathbf{q}\cdot\mathbf{r}_{21})] G_2(1, 2; t) \tag{A.2}$$

Then we have

$$\theta_{12} = \int d\mathbf{q}\, [\exp(i\mathbf{q}\cdot\mathbf{r}_{21})]\, d_{12}(\mathbf{q}) \tag{A.3}$$

where $d_{12}(\mathbf{q}) \equiv -V_q i\mathbf{q}\cdot[(\partial/\partial\mathbf{p}_1) - (\partial/\partial\mathbf{p}_2)]$, and (A.1) leads to

$$\begin{aligned} K_1(\mathbf{q}) = &\int d\mathbf{p}_2\, \delta_-^{12}(\mathbf{q})\, d_{21}(\mathbf{q}) f(1;t) f(\mathbf{p}_2, \mathbf{r}_1; t) \\ &+ c\int d\mathbf{p}_2\, \delta_-^{12}(\mathbf{q})\, [d_2(\mathbf{q}) f(\mathbf{p}_2, \mathbf{r}_1; t) K_1(\mathbf{q}) \\ &+ d_1{}^*(\mathbf{q}) f(1;t) K^*_{\mathbf{p}_2\mathbf{r}_1}(\mathbf{q})] \end{aligned} \tag{A.4}$$

where $d_1(\mathbf{q}) \equiv -(2\pi)^3 V_q i\mathbf{q}\cdot(\partial/\partial\mathbf{p}_1)$, $\delta_-^{12}(\mathbf{q}) \equiv \pi\delta_-(\mathbf{q}\cdot\mathbf{g}_{12})$, and $\mathscr{X}^*$ denotes the complex conjugate of $\mathscr{X}$. Since the scaling (13) leads to

$$\int_a^\infty dr_{21}\, r_{21}^2 \int d\omega \int d\mathbf{p}_2\, \theta_{12} G_2(1,2;t) \to \int d(2)\, \theta_{12} G_2(1,2;t) \tag{A.5}$$

we have $BL_1(f) = \int d\mathbf{q}\, d_1(\mathbf{q}) K_1(\mathbf{q})$. By solving (A.4) for $K_1(\mathbf{q})$, we can easily transform $BL_1(f)$ into the Balescu–Lenard collision term (31).[16,17]

APPENDIX B. DERIVATION OF (38)

Applying the scaling (13) and $\mathbf{r}_{21} \to L\mathbf{r}_{21}$ to (12), we obtain

$$\begin{aligned} &H_2(1,2,3;t) \\ &\to L^{-5/2} \int_0^\infty ds\, \exp(-s\mathbf{g}_{31}\cdot\partial/\partial\mathbf{r}_{31}) \\ &\quad\times (\theta_{13} f(1;t)[\chi(2;\mathbf{p}_3,\mathbf{r}_1;t) \\ &\quad- (1/c)\,\delta(\mathbf{p}_2 - \mathbf{p}_3)\,\delta(\mathbf{r}_2 - \mathbf{r}_1) f(2;t)] \\ &\quad+ \theta_{13} f(\mathbf{p}_3,\mathbf{r}_1;t)[\chi(1,2;t) - (1/c)\,\delta(1-2) f(1;t)] \\ &\quad+ c\int d(4)\,\{\theta_{14} f(1;t) H_2(2,\mathbf{p}_3,\mathbf{p}_4,\mathbf{r}_1,\mathbf{r}_{34};t) \\ &\quad+ \theta_{34} f(\mathbf{p}_3,\mathbf{r}_1;t) H_2(1,2,\mathbf{p}_4,\mathbf{r}_{41};t) + \theta_{34} G_2(1,\mathbf{p}_4,\mathbf{r}_{41};t) \\ &\quad\times [\chi(2,\mathbf{p}_3,\mathbf{r}_1;t) - (1/c)\,\delta(\mathbf{p}_2 - \mathbf{p}_3)\,\delta(\mathbf{r}_2 - \mathbf{r}_1) f(2;t)] \\ &\quad+ \theta_{14} G_2(\mathbf{p}_3,\mathbf{p}_4,\mathbf{r}_1,\mathbf{r}_{43};t) \\ &\quad\times [\chi(1,2;t) - (1/c)\,\delta(1-2) f(1;t)]\}) \end{aligned} \tag{B.1}$$

and $\nu_2 = 5$, where we have used the fact that $\int d\mathbf{r}_{42}\, \theta_{42} = 0$. Let us define

$$\begin{aligned} L_{1,2}(\mathbf{q}) &\equiv L_{\mathbf{p}_1\mathbf{r}_1,\mathbf{p}_2\mathbf{r}_2}(\mathbf{q}) \\ &\equiv (2\pi)^{-3} \int d(3)\, [\exp(i\mathbf{q}\cdot\mathbf{r}_{31})] H_2(1,2,3;t) \end{aligned} \tag{B.2}$$

Then (B.1) leads to

$$
\begin{aligned}
L_{1,2}(\mathbf{q}) = \int d\mathbf{p}_3\, \delta_-^{13}(\mathbf{q})\{d_{31}(\mathbf{q})f(1;t)[\chi(2,\mathbf{p}_3,\mathbf{r}_1;t) \\
- (1/c)\,\delta(\mathbf{p}_2 - \mathbf{p}_3)\,\delta(\mathbf{r}_2 - \mathbf{r}_1)f(2;t)] \\
+ d_{31}(\mathbf{q})f(\mathbf{p}_3,\mathbf{r}_1;t)[\chi(1,2;t) - (1/c)\,\delta(1-2)f(1;t)] \\
+ c[d_1{}^*(\mathbf{q})f(1;t)L^*_{\mathbf{p}_3,\mathbf{r}_1,2}(q) + d_3(\mathbf{q})f(\mathbf{p}_3,\mathbf{r}_1;t)L_{1,2}(\mathbf{q})] \\
+ c\, d_3(\mathbf{q})K_1(\mathbf{q})[\chi(2,\mathbf{p}_3,\mathbf{r}_1;t) \\
- (1/c)\,\delta(\mathbf{p}_2 - \mathbf{p}_3)\,\delta(\mathbf{r}_2 - \mathbf{r}_1)f(2;t)] \\
+ c\, d_1{}^*(\mathbf{q})K^*_{\mathbf{p}_3,\mathbf{r}_1}(\mathbf{q})[\chi(1,2;t) - (1/c)\,\delta(1-2)f(1;t)]\}
\end{aligned} \tag{B.3}
$$

It can be shown from (A.4) that $\int d(3)\,[\delta K_1(\mathbf{q})/\partial f(3;t)](\chi(3,2;t) - (1/c)\,\delta(3-2)f(2;t)]$ is a solution of (B.3). Since the scaling (13) and $\mathbf{r}_{21} \to L\mathbf{r}_{21}$ lead to

$$
\begin{aligned}
\int_a^\infty dr_{31}\, r_{31}^2 \int d\omega \int d\mathbf{p}_3\, \theta_{13}H_2(1,2,3;t) \\
\to L^{-2}\int d(3)\,\theta_{13}H_2(1,2,3;t) = L^{-2}\int d\mathbf{q}\, d_1(\mathbf{q})L_{1,2}(\mathbf{q})
\end{aligned} \tag{B.4}
$$

we thus obtain (38).

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